Solute transport through heterogeneous porous medium with effective adsorption coefficient

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ABSTRACT

One-dimensional solute transport, originating from a continuous point source, is studied along unsteady longitudinal flow through a heterogeneous medium of semi-infinite extent. Diffusion is considered as directly proportional to the linear spatially–dependent function that defines the heterogeneity. It is also assumed temporally dependent. It is expressed in both the independent variables in degenerate form. The adsorption parameter is considered to be inversely proportional to diffusion coefficient. Certain new independent variables are introduced through separate transformations to reduce the variable coefficients of the advection-diffusion equation to constant coefficients. The Laplace Transformation Technique (LTT) is used to obtain the desired solution. The effects of adsorption, heterogeneity and unsteadiness on the solute transport are investigated with various graphs.

Key words: advection; adsorption; anisotropic; diffusion; heterogeneous; porous medium.

1 INTRODUCTION

An 'adsorption' is a process in which solute concentration in the liquid phase is attracted by the solid boundaries of the pores and is deposited on them (a process in which molecules accumulate in the interfacial layer). In present time and in the coming future, the adsorption process requires approximate theoretical descriptions for definite and realistic models and will still be of great interest in the studies of adsorption at the solid-fluid interface. Due to complex structure of porous medium that porous consists of pores of different sizes and shapes. The significance of pores in the adsorption processes largely depends on their shapes and sizes. A theory of the monomolecular adsorption on energetically homogeneous surfaces is first given by Langmuir and latter tried to extend theoretical approach to account for heterogeneity of solid adsorbent and the multilayer character of adsorption. One fundamental assumptions of the theory which refers to the homogeneity of the adsorbent surface is not justified in many cases. In case of heterogeneous, the adsorption sites are distributed over energetically different levels. Such studies are of great importance in the remedial processes of groundwater, soil, industrial and blood where substances are in mixture form as well as in the environmental protection processes (Dabrowski, A., 2001).

The study of hydro-dynamic dispersion process is important in water quality management particularly in aquifers and environmental protection from pollutants, the dispersion has been referred to as a hydraulic mixing process by which the waste concentrations are attenuated while the waste pollutants are being transported downstream. The concentration distribution behavior with space and time is described by a partial differential equation of parabolic type known as advection–diffusion equation

(ADE). A one dimensional ADE, derived on the principle of conservation of mass and Fick's law of diffusion in general form may be written as:

$$\frac{\partial C}{\partial t}+\frac{1-n_p}{n_p}\frac{\partial F}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x}-u(x,t)C\right) \quad (1)$$

where C is the solute concentration at position x at time t in liquid phase and F is the concentration in the solid phase at time t, n_p is porosity. $D(x,t)$ and $u(x,t)$ are diffusion coefficient and velocity of the flow field, respectively. In Eq. (1), D and u may be constants or functions of independent variables. This equation is solved in which one or both the coefficients either functions of independent variables or both constant. Many analytical solutions of the advection–diffusion equations (ADE) have been obtained subject to variety of initial and boundary conditions. We have reviewed some important hydro-dynamic dispersion problems based on theories of Scheidegger (1957), Ebach and White (1958), Rumer (1962), Scheidegger (1957), Matheron and deMarsily (1980) and tackled them analytically (Jaiswal et al. 2009, 2011a,b) and Kumar et al. (2010).

In previous literatures, the longitudinal dispersion coefficient is linearly and squarely proportional to the fluid velocity. Ogata and Bank (1961), Banks and Jerasate (1962), Lin (1977), Al-Niami and Rushton (1977) obtained analytical solutions for dispersion problems in a porous medium. Several such type of works have compiled by van Genuchten and Alves (1982) and Lindstrom and Boersma (1989). Tracy (1995) first gave some simple one-dimensional solutions. Next, by use of a transformation, the nonlinear partial differential equation is converted to a linear one for a specific from of the moisture content vs. Pressure head and relative hydraulic conductivity vs. Pressure head curves. This allows both two and three dimensional solutions to be derived. Aral and Liao (1996) examined solutions to two dimensional advection-dispersion equation with time dependent dispersion coefficients and demonstrated the time and space dependent nature of the dispersion coefficient in subsurface contaminant transport problems.

Nowadays, some dispersion problems dealing with heterogeneous media are reviewed. Liu et al. (2000) applied the generalized integral transform technique to solve the one-dimensional advection-dispersion equation in heterogeneous porous media coupled with either linear or nonlinear sorption or decay, and with spatially and temporally variable flow and dispersion coefficient, and arbitrary initial and boundary conditions. Sander and Braddock (2005) presented a range of analytical solutions for combined transient water and solute transport for horizontal flow adopting the concept of a scale and time dependent dispersivity for contaminant transport in aquifers. They applied these solutions to the transient, unsaturated horizontal flow to develop similarity solutions for both constant solute concentration and solute flux boundary conditions. Kartha and Srivastava (2008) studied the effect of immobile water content on contaminant advection and dispersion in unsaturated porous media. Guerrero et al. (2009) presented an analytical methodology by using change-of-variables in combination with GITT, to solve the advection-diffusion equation in a finite domain for both transient and steady-state regimes. Cassol et al. (2009) presented an analytical solution for the two-dimensional atmospheric pollutant dispersion problem utilizing GITT, the Laplace Transform and matrix diagonalization. Moreira et al. (2009) presented a review of the GILTT solutions focusing the applications to pollutant dispersion in atmosphere. Jaiswal et al. (2012, 2013, 2014) and Yadav et al. (2010, 2012a,b) obtained an analytical solutions with the help of Laplace transform technique (LTT) in finite and semi-infinite domain for dispersion problems with different boundary conditions related to physical and real scenario.

In the present paper, diffusion parameter is inversely proportional to the adsorption coefficient. The two coefficients, diffusion and adsorption of the ADE are considered as functions of both the independent variables (space and time variable). The temporal dependence is due to unsteadiness of the flow medium and the spatial dependence is due to the heterogeneity of the medium. An introduction of new independent space and time variables, the ADE with variable coefficients is reduced into constant coefficients.

2 MATHEMATICAL FORMULATION

The linear advection-diffusion partial differential equation in one dimension in general form defined in Eq. (1). This equation is solved for a dispersion problem in which both the coefficients remain functions of independent variables. In Eq. (1), D and u may be constants or functions of independent variables. Lapidus and Amundson (1952) considered two cases, namely,

$$F = k_1 C^n + k_2 \tag{2}$$

and

$$\frac{\partial F}{\partial t} = k_1 C^n - k_2 \tag{3}$$

Equilibrium and non-equilibrium isotherm between the concentrations in the two phases, where k_1 and k_2 are empirical constants of the medium. The isotherm is linear if $n = 1$, and is non-linear if $n > 1$. For simplicity, the former relationship is adopted in the present analysis. The use of equilibrium isotherms assumes that equilibrium exists at all times between the porous medium and the solute in solution. This assumption is generally valid when the adsorption process is fast in relation to the ground-water velocity (Cherry et al., 1984). Using Eq. (2) in Eq. (1) for $n = 1$ we may get,

$$\frac{\partial C}{\partial t} + \frac{1-n_p}{n_p}\frac{\partial (k_1 C + k_2)}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x} - u(x,t)C \right) \tag{4}$$

or

$$R(x,t)\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x} - u(x,t)C \right) \tag{5}$$

where $R(x,t) = 1 + \frac{1-n_p}{n_p}k_1$, is the adsorption coefficient. The fundamental concept in adsorption science is that named as the adsorption isotherm. It is the equilibrium relation between the quantity of the adsorbed material and the pressure or concentration in the bulk fluid phase at constant temperature. The diffusion and adsorption parameter is considered inversely proportional to each other. It means when adsorption is higher, then diffusion processes is slow and vice-versa. It is show in real and physical situations. Thus we write

$$D(x,t) \propto \frac{1}{R(x,t)} \quad \text{or} \quad R(x,t) \propto \frac{1}{D(x,t)}$$

The medium is considered to be of semi-infinite extent anisotropic and heterogeneous. In an anisotropic medium direction changes with time and in a heterogeneous porous medium, porosity changes with position. As flow velocity depends upon both i.e. time and position, so it is non-uniform. In a non-porous medium, such as air or a surface water body, velocity is rarely uniform. Let velocity at the origin $x = 0$ and in entire domain be temporally dependent. Since in porous domain like as, soil, blood, aquifers and groundwater, the velocity is changed with time. As a consequence of the heterogeneity of the medium, the diffusion coefficient transporting the solute particles "spread out" is considered spatially dependent.

Its expression of increasing nature is linearly interpolated in the position variable in a finite longitudinal region $0 \le x \le \ell$. Let diffusion coefficient at the origin $x = 0$ of the domain be D_0, which increases to $D_0(1+b)$ at $x = \ell$, where a real constant $b < 1$ it is clear that the change in diffusion coefficient is of small order, i.e. the laminar condition of the flow is not affected. Thus the expression for diffusion coefficient at position x, may be linearly interpolated as:

$$D(x) = \frac{x-\ell}{0-\ell}D_0 + \frac{x-0}{\ell-0}D_0(1+b) = D_0(1+ax) \tag{6}$$

where $a = b/\ell$ is a constant less than 1.0 and serves as a parameter of heterogeneity of the medium. Its different values represent media of varying heterogeneity. Further, the diffusion is also considered temporally dependent. The expression for dispersion is written in degenerate form as:

$$D(x,t) = D_0 f(mt)(1+ax)$$

(7)

where m may be termed as an unsteady parameter of dimension inverse of the dimension of t. While choosing an expression for $f(mt)$, it is ensured that $f(mt) = 1$ for $m = 0$ and $t = 0$. The former case represents the steady flow. The latter case represents the velocity at the initial stage. Therefore the adsorption parameter and velocity may be written as

$$R(x,t) = \frac{R_0}{f(mt)(1+ax)}$$

(8)

and $u(x,t) = u_0 f(mt)$, then the equation (1) becomes,

$$\frac{R_0}{f(mt)(1+ax)}\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f(mt)(1+ax)\frac{\partial C}{\partial x} - u_0 f(mt)C \right)$$

(9)

introduce a new independent variable X, using a transformation which is

$$X = -\int \frac{1}{(1+ax)^2}dx$$

(10)

As ax and mt are non-dimensional terms, so the constants u_0 and D_0 in equation (9) may be referred to as uniform velocity of dimension LT^{-1} and the initial diffusion coefficient of dimension L^2T^{-1}, respectively. Using equation (10), equation (9) may be written as

$$\frac{R_0}{f^2(mt)}\frac{\partial C}{\partial t} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + \left(a^2 D_0 + a u_0\right) X \frac{\partial C}{\partial X}$$

(11)

where $$X = \frac{1}{a(1+ax)}$$.

(12)

Again using a transformation (Crank, 1975)

$$T = \int_0^t \frac{f^2(mt)}{R_0}dt$$

(13)

The advection-diffusion equation (11) becomes

$$\frac{\partial C}{\partial T} = a^2 D_0 X^2 \frac{\partial^2 C}{\partial X^2} + \left(a^2 D_0 + a u_0\right) X \frac{\partial C}{\partial X}$$

(14)

Now with the help of other independent variables introduced through the transformation

$$Z = -\ln aX \quad , \tag{15}$$

the variable coefficients of the advection-diffusion equation are reduced to constant coefficients.

$$\frac{\partial C}{\partial T} = a^2 D_0 \frac{\partial^2 C}{\partial Z^2} - au_0 \frac{\partial C}{\partial Z} \quad , \quad (0 \le Z < \infty \,, \quad T > 0) \tag{16}$$

Also an expression of $f(mt)$ is chosen such that for $t = 0$, we get $T = 0$, so that the nature of initial condition does not change. To solve equation (9), one initial condition and two boundary conditions are required.

3 ANALYTICAL SOLUTIONS

3.1 Uniform continuous point source condition

It is assumed that initially the medium is solute free. The source of the solute mass is a uniform continuous point source at the origin. It serves as the first boundary condition known as the input condition. A flux type homogeneous condition is assumed at the far end of the medium.

Thus initial condition is

$$C(x,t) = 0 \quad (t = 0\,, x \ge 0) \tag{17}$$

the input and second boundary condition are

$$C(x,t) = C_0 \quad (x = 0\,, \quad t > 0) \tag{18}$$

$$\frac{\partial C}{\partial x} = 0 \quad (x \to \infty, \quad t \ge 0) \tag{19}$$

The initial and boundary conditions (equations (17)-(19)) may be written in the (Z,T) domain as,

$$C(Z,T) = 0 \quad (T = 0, \quad Z \ge 0) \tag{20}$$

$$C(Z,T) = C_0 \quad (Z = 0, \quad T > 0) \tag{21}$$

$$\frac{\partial C}{\partial Z} = 0 \quad (Z \to \infty, \; T \ge 0) \tag{22}$$

Now the Laplace transform may be used to get the analytical solution. But to use it more conveniently, the convective term in the advection-diffusion equation (16) is eliminated by a transformation:

$$C(Z,T) = K(Z,T)\exp\left(\frac{u_0}{2aD_0} Z - \frac{u_0^2}{4D_0} T\right) \tag{23}$$

Thus the initial and boundary value problem defined by equations (16), (20)-(22) may be written in terms of a new dependent variable, $K(Z,T)$ as,

$$\frac{\partial K}{\partial T} = a^2 D_0 \frac{\partial^2 K}{\partial Z^2} \tag{24}$$

$$K(Z,T) = 0 \quad (T = 0\,, Z \ge 0) \tag{25}$$

$$K(Z,T) = C_0 \exp\left(\beta^2 T\right) \qquad (Z = 0, \qquad T > 0), \qquad \text{where } \beta^2 = \frac{u_0^2}{4D_0}$$

(26)

$$\frac{\partial K}{\partial Z} = -\frac{u_0}{2aD_0} K \qquad (Z \to \infty, \ T \geq 0)$$

(27)

Now to eliminate time variable from partial differential equation (24), *i.e.*, to reduce it into an ordinary differential equation of second order, let us multiply both sides of (24) by $\exp(-pT)$ and integrate it between $T = 0$ and $T \to \infty$, *i.e.*, applying Laplace transformation of parameter p, on equation (24) we get

$$\int_0^\infty \frac{\partial K}{\partial T} \exp(-pT) dT = \int_0^\infty a^2 D_0 \exp(-pT) \frac{\partial^2 K}{\partial Z^2} dT$$

or $$\left[K(Z,T)\exp(-pT)\right]_0^\infty + p\bar{K} = a^2 D_0 \frac{d^2 \bar{K}}{dZ^2}$$

(28)

where $$\bar{K}(Z,p) = \int_0^\infty K(Z,T) \exp(-pT) dT$$

(29)

Now using initial condition (25), above equation becomes

$$\frac{d^2 \bar{K}}{dZ^2} - \frac{p}{a^2 D_0} \bar{K} = 0$$

(30)

The input boundary conditions (26) becomes as

$$\int_0^\infty K(Z,T) \exp(-pT) dT = \int_0^\infty C_0 \exp\left(\frac{u_0{}^2}{4D_0} T\right) \exp(-pT) dT$$

or $$\bar{K}(Z,p) = \frac{C_0}{(p - \beta^2)} \qquad \text{at} \qquad Z = 0, \qquad \text{where} \qquad \beta^2 = u_0{}^2 / 4D_0$$

(31)

and second boundary condition (27) becomes

$$\int_0^\infty \frac{\partial K(Z,T)}{\partial Z} \exp(-pT) dT = -\int_0^\infty \frac{u_0}{2aD_0} K(Z,T) \exp(-pT) dT$$

or $$\frac{d\bar{K}(Z,p)}{dZ} = -\frac{u_0}{2D_0} \bar{K}(Z,p) \qquad \text{at} \qquad Z \to \infty$$

(32)

The general solution of differential equation (30) may be written as

$$\bar{K}(Z,p) = c_1 \exp\left(-Z\sqrt{p / a^2 D_0}\right) + c_2 \exp\left(Z\sqrt{p / a^2 D_0}\right)$$

(33)

where c_1 and c_2 are arbitrary constants. Using condition (32) on the above solution we get $c_2 = 0$. So general solution (33) becomes

$$\bar{K}(Z,p) = c_1 \exp\left(-Z\sqrt{p / a^2 D_0}\right)$$

(34)

Now using input condition (31) on above solution we get

$$c_1 = \frac{C_0}{(p - \beta^2)}$$

Thus the particular solution in the Laplacian domain may be written as

$$\bar{K}(Z,p)=\frac{C_0}{(p-\beta^2)}\exp-\left(Z\sqrt{p/a^2D_0}\right)$$

(35)

Now taking Inverse Laplace transform of equation (35), the solution in $K(Z,T)$ is obtained. For this purpose the table from van Genutchen and Alves (1982) has been used. The solution in $K(Z,T)$ may be written as

$$K(Z,T)=\frac{C_0}{2}\left[\exp\left\{\beta^2T-\beta Z\sqrt{\frac{1}{a^2D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{a^2D_0T}}-\beta\sqrt{T}\right\}\right.$$
$$\left.+\exp\left\{\beta^2T+\beta Z\sqrt{\frac{1}{a^2D_0}}\right\}erfc\left\{\frac{Z}{2\sqrt{a^2D_0T}}+\beta\sqrt{T}\right\}\right] \quad (36)$$

using back transformations Eqs. (23), (15), (13) and (12), as follows

$$C(x,T)=\frac{C_0}{2}\left[\exp\left\{\left(\frac{u_0}{2aD_0}-\frac{\beta}{\sqrt{a^2D_0}}\right)\ln(1+ax)\right\}erfc\left\{\frac{\ln(1+ax)}{2\sqrt{a^2D_0T}}-\beta\sqrt{T}\right\}\right.$$
$$\left.+\exp\left\{\left(\frac{u_0}{2aD_0}+\frac{\beta}{\sqrt{a^2D_0}}\right)\ln(1+ax)\right\}erfc\left\{\frac{\ln(1+ax)}{2\sqrt{a^2D_0T}}+\beta\sqrt{T}\right\}\right]$$

(37)

where

, $T=\int_0^t\frac{f^2(mt)}{R_0}dt$ and $\beta^2=\frac{u_0^2}{4D_0}$.

3.2 Increasing input point source condition

The input point source condition defined by (18) is of uniform nature. It means in the presence of the source the input concentration constant for all time. But in real cases the infiltration of pollutants from the earth surface into an aquifer and solute transport in other porous medium may increase due to increasing human and other responsible activities. This more realistic scenario may be defined by a condition of mixed type of non-homogeneous nature which is

$$-D(x,t)\frac{\partial C(x,t)}{\partial x}+u(x,t)C(x,t)=u_0C_0 \quad (x=0,\ t>0\quad)$$

(38)

the above Eq. (38) may be written in $K(Z,T)$ domain as

$$-\frac{\partial K(Z,T)}{\partial Z}+\frac{u_0}{2aD_0}K(Z,T)=\frac{u_0C_0}{aD_0f(mt)}\exp\{\beta^2T\}\ ; \qquad Z=0$$

(39)

Now to proceed further, expression for $f(mt)$ has to be chosen. Let the expression be of exponentially decreasing nature. Let

$$f(mt)=\exp(-mt)$$

(40)

From equation (13), we get the expression for new time variable as

$$T=\frac{1}{2mR_0}\left[1-\exp(-2mt)\right],$$

(41)

From the above expression of T, we may write $f(mt)$ in terms of T as follows

$$f\left(mt\right)=\exp(-mt)=\left(1-2mR_0T\right)^{1/2}$$

(42)

So condition (39) becomes

$$-\frac{\partial K(Z,T)}{\partial Z}+\frac{u_0}{2aD_0}K(Z,T)=\frac{u_0C_0}{aD_0}\left(1+mR_0T\right)\exp\{\beta^2T\}$$

where the series $o(m^2)$ from the binomial expansion is neglected as m is chosen much smaller than 1.0. Applying Laplace Transform Technique, above condition becomes

$$-aD_0\frac{d\bar{K}}{dZ}+\frac{u_0}{2}\bar{K}=\frac{u_0C_0}{(p-\beta^2)}+\frac{mu_0R_0C_0}{(p-\beta^2)^2}$$

(43)

Now using the condition Eq. (43) in place of Eq. (31) the arbitrary constant c_1 from general solution Eq. (34) may be obtained as

$$c_1=\frac{u_0C_0}{\sqrt{D_0}(p-\beta^2)(\sqrt{p}+\beta)}+\frac{mu_0R_0C_0}{\sqrt{D_0}(p-\beta^2)^2(\sqrt{p}+\beta)}$$

(44)

Thus the general solution (30) for increasing input source condition is,

$$\bar{K}(Z,p)=\frac{u_0C_0\exp-\left(Z\sqrt{p/a^2D_0}\right)}{\sqrt{D_0}(p-\beta^2)(\sqrt{p}+\beta)}+\frac{mu_0R_0C_0\exp-\left(Z\sqrt{p/a^2D_0}\right)}{\sqrt{D_0}(p-\beta^2)^2(\sqrt{p}+\beta)}$$

(45)

Now taking Inverse Laplace transform of equation (45), the solution in $K(Z,T)$ is obtained. For this purpose the table from van Genutchen and Alves (1982) is used. Thus the solution in $K(Z,T)$ may be derived as

$$\begin{aligned}K(Z,T)=\frac{u_0C_0}{\sqrt{D_0}}&\left[\frac{1}{4\beta}\exp\left(\beta^2T-\beta Z/\sqrt{a^2D_0}\right)\operatorname{erfc}\left(\frac{Z/\sqrt{a^2D_0}}{2\sqrt{T}}-\beta\sqrt{T}\right)\right.\\&-\frac{1}{4\beta}\left(1+2\beta Z/\sqrt{a^2D_0}+4\beta^2T\right)\exp\left(\beta^2T+\beta Z/\sqrt{a^2D_0}\right)\times\\&\left.\operatorname{erfc}\left(\frac{Z/\sqrt{a^2D_0}}{2\sqrt{T}}+\beta\sqrt{T}\right)+\sqrt{\frac{T}{\pi}}\exp\left\{\frac{\left(-Z/\sqrt{a^2D_0}\right)^2}{4T}\right\}\right]\\+\frac{mu_0R_0C_0}{\sqrt{D_0}}&\left[\frac{T}{4\beta^2}(1+\beta Z/\sqrt{a^2D_0}+2\beta^2T)\exp\left\{\frac{-\left(Z/\sqrt{a^2D_0}\right)^2}{4T}\right\}\right.\\&+\frac{1}{16\beta^3}\left(4\beta^2T-2\beta Z/\sqrt{a^2D_0}-1\right)\\&\exp\left(\beta^2T-\beta Z/\sqrt{a^2D_0}\right)\operatorname{erfc}\left(\frac{Z/\sqrt{a^2D_0}}{2\sqrt{T}}-\beta\sqrt{T}\right)\\&-\frac{1}{16\beta^3}\left\{4\beta^2T-1+2\beta^2(Z/\sqrt{a^2D_0}+2\beta T)^2\right\}\\&\left.\exp\left(\beta^2T+\beta Z/\sqrt{a^2D_0}\right)\operatorname{erfc}\left(\frac{Z/\sqrt{a^2D_0}}{2\sqrt{T}}+\beta\sqrt{T}\right)\right]\end{aligned}$$

(46)

Using back transformations Eqs. (23), (15) (13) and (12), to get the desired solution of advection-diffusion equation for input point source for increasing nature may be written in terms of $C(X,T)$ as follows

$$
\begin{aligned}
C(x,T) = \frac{C_0}{2}\Bigg[& 2u_0\sqrt{\frac{T}{\pi D_0}}\exp\left(\frac{u_0\ln(1+ax)}{2aD_0}-\frac{u_0^2T}{4D_0}-\frac{\ln(1+ax)^2}{4a^2D_0T}\right)+\operatorname{erfc}\left(\frac{\ln(1+ax)-au_0T}{2a\sqrt{D_0T}}\right) \\
& -\left(1+\frac{u_0\ln(1+ax)}{aD_0}+\frac{u_0^2T}{D_0}\right)\exp\left(\frac{u_0\ln(1+ax)}{aD_0}\right)\operatorname{erfc}\left(\frac{\ln(1+ax)+au_0T}{2a\sqrt{D_0T}}\right) \\
& +\frac{mD_0R_0}{u_0^2}\Bigg\{2u_0\sqrt{\frac{T}{\pi D_0}}\left(1+\frac{u_0\ln(1+ax)}{2aD_0}+\frac{u_0^2T}{2D_0}\right)\exp\left(\frac{u_0\ln(1+ax)}{2aD_0}-\frac{u_0^2T}{4D_0}-\frac{\ln(1+ax)^2}{4a^2D_0T}\right) \\
& -\left(1+\frac{u_0\ln(1+ax)}{aD_0}-\frac{u_0^2T}{D_0}\right)\operatorname{erfc}\left(\frac{\ln(1+ax)-au_0T}{2a\sqrt{D_0T}}\right) \\
& +\left(1-\frac{u_0^2T}{D_0}-\frac{u_0^2}{2a^2D_0^2}\{\ln(1+ax)+au_0T\}^2\right) \\
& \times\exp\left(\frac{u_0\ln(1+ax)}{aD_0}\right)\operatorname{erfc}\left(\frac{\ln(1+ax)+au_0T}{2a\sqrt{D_0T}}\right)\Bigg\}\Bigg]
\end{aligned}
\tag{47}
$$

where $T=\frac{1}{2mR_0}\left[1-\exp(-2mt)\right]$.

4 RESULTS AND DISCUSSION

The two coefficients, diffusion and adsorption of the ADE are considered as functions of both the independent variables (space and time variable) i.e., $D(x,t)=D_0f(mt)(1+ax)$ and $R(x,t)=\frac{R_0}{f(mt)(1+ax)}$ while the flow velocity $\{u(x,t)=u_0f(mt)\}$ is considered temporally dependent. The temporal dependence is due to unsteadiness of the flow medium and the spatial dependence is due to the heterogeneity of the medium. The Laplace transform technique (LTT) which is more viable than other methods. By using LTT, in the present paper, an analytical solution is obtained for a one-dimensional advection-diffusion equation (ADE) describing the solute transport in a realistic scenario. The solution may be used more effectively than previous ones to construct the mass transport function for a new type of transient infinite element (Zhao and Valliappan 1994, Zhao 2009), and other numerical solutions in a semi-infinite domain (van Genuchten et al. 2013a,b).

Concentration values are evaluated from analytical solution (37) and (47) for $C_0=1.0$ initial velocity $u_0=0.61$ (km/yr), and initial diffusion coefficient $D_0=0.73$ (km²/yr). The concentration values (C/C_0) are evaluated in a finite domain $0\le x\le 10$, of the semi-infinite medium. Exponentially accelerating and decelerating forms of $f(mt)$ may be considered. All graphs are depicting the concentration distribution in the assumed domain at $t(\text{yr})=4,7$ for the form $u=u_0\exp(-mt)$. It ensures the Fick's law (on which ADE is based) holds good in with the heterogeneity of the medium and unsteadiness of the flow field. In all figures, horizontal axis shows position while vertical axis represents the concentration (C/C_0).

For $f(mt)=\exp(-mt)$, the concentrations evaluated from solution (37) and (47) for $t\,(\text{yr})=4,7$ with heterogeneity parameter $a=0.1\ (\text{km})^{-1}$, $R_0=1.25$ and unsteady flow parameter $m=0.1\ (\text{yr})^{-1}$ are depicted by two solid and thick curves for uniform and increasing input source condition in Fig. 1 and Fig 5 respectively. The solute transport behaves as expected, i.e. the concentration values decrease with distance and increase with time in uniform nature but it converse for increasing input condition. Comparisons have been done for various adsorption coefficient, heterogeneity parameter and unsteady parameter with same initial data and show in the graphs separately. Figures 2 and 6 show by thick dotted the effect of adsorption coefficient on solute transport along an exponentially decelerating flow field at $t = 4$ yr. The concentration values are evaluated from solution (37) and (47) for $R_0 = 1.25$ and 1.75. It may be observed that at a particular position, concentration decreases with the adsorption parameter. In Figs 3 and 7, the effect of heterogeneity on solute transport along an exponentially decelerating flow field, solute concentration represented by thin solid curves at $t = 4$ yr. The concentration values are evaluated from solution (37) and (47) for $a\ (\text{km})^{-1} =$ 0.1 and 0.2. It may be observed that at a particular position, concentration increases with the heterogeneity parameter. In other words, solute transport is faster in a medium of higher heterogeneity of increasing nature than that in a medium of lower heterogeneity. This is because (u/D) at a position increases with a. Figures 4 and 8 show the effect of unsteady parameter along flow field at $t = 4$ yr from solution (37) and (47) for $m\ (yr)^{-1} = 0.1$ and 0.15. It may be observed that the concentration decreases with the unsteady parameter at a particular position.

Analytical solution for a physical aspect is of fundamental importance to understanding the role of all the parameters in the physical phenomenon. In real scenarios the parameters are more complex and obtaining an analytical solution in much difficult, while numerical solution is an alternative (possible due to the advent of fast computing). But numerical solutions require analytical solution of a similar less general problem for validation in terms of convergence and stability.

5 CONCLUSIONS

In the present study, diffusion coefficient is considered in degenerate form i.e., as directly proportional to the linear spatially–dependent function that defines the heterogeneity and temporally dependent that define unsteadiness. It is expressed in both the independent variables. The adsorption parameter is considered to be inversely proportional to diffusion coefficient and the flow velocity is considered temporally dependent. The Laplace Transformation Technique (LTT) is used to obtain the desired solution with introducing certain new independent variables to reduce the variable coefficients of the advection-diffusion equation into constant coefficients. The effects of adsorption, heterogeneity and unsteadiness on the solute transport are shows with various graphs.

The analytical solutions of ADE with effective adsorption coefficient have wide applications in water, environment, industries etc along with heterogeneity and unsteadiness and validate the numerical solutions. Some of the most widespread uses of activated carbons for liquid-phase adsorption are those in water treatment. Recent years have seen remarkable increase in the level of synthetic organic chemicals (SOC) in public water supplies. More SOCs such as pesticides, herbicides, detergents, polycyclic aromatic hydrocarbons, nitrosamines, phenolic compounds, trihalomethanes and other pollutants, have been identified in drinking water supplies. On the other hand, natural organic material NOM. is found in varying concentrations in all natural water sources. It is a complex mixture of compounds formed as a result of decomposition of animal and plant materials in the environment. As a consequence, the use of activated carbons in water treatment has increased throughout the world.Every aspect of human activity is closely connected with the natural environment. Whether or not we are aware or care each of us interacts with and affects our environment every day.

ACKNOWLEDGEMENTS

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FIGURES

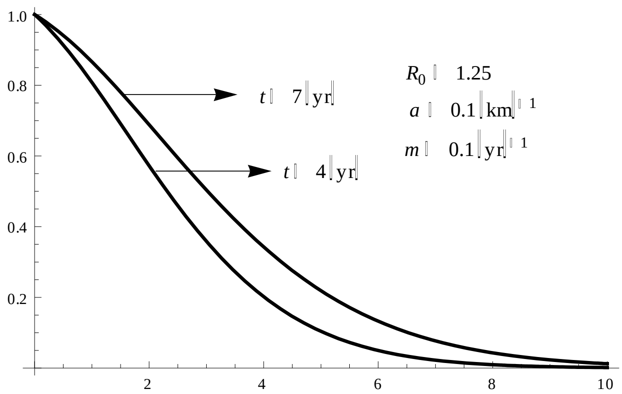

Fig 1. Solute transport described by the solution (37) where $m = 0.1(\text{year})^{-1}$, $a = 0.1(\text{km})^{-1}$ and $R_0 = 1.25$ for $t(\text{yr}) = 4, 7$

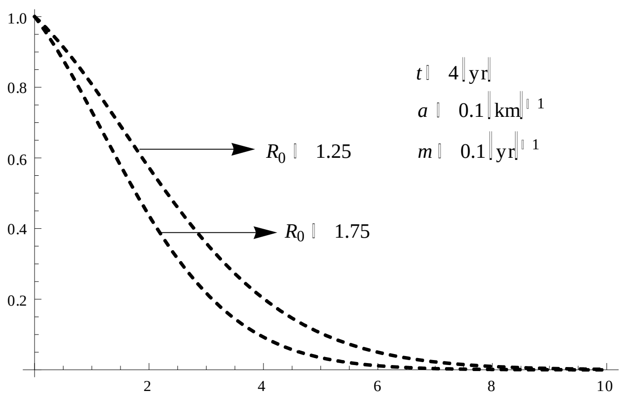

Fig 2. Solute transport described by the solution (37) where $m = 0.1(\text{year})^{-1}$, $t(\text{yr}) = 4$ and $a = 0.1(\text{km})^{-1}$ for $R_0 = 1.25, 1.75$

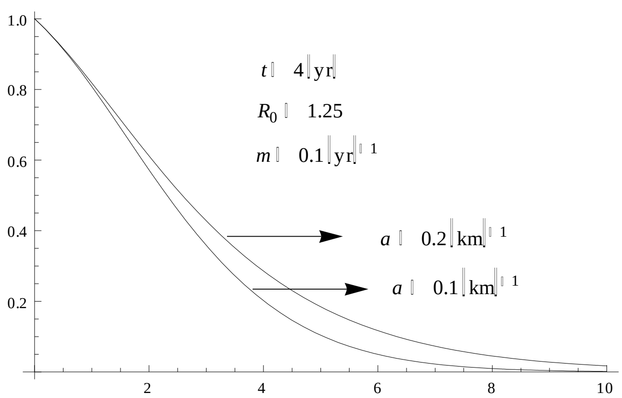

Fig 3. Solute transport described by the solution (37) where $m = 0.1(\text{year})^{-1}$, $t(\text{yr}) = 4$ and $R_0 = 1.25$ for $a = 0.1, 0.2\ (\text{km})^{-1}$

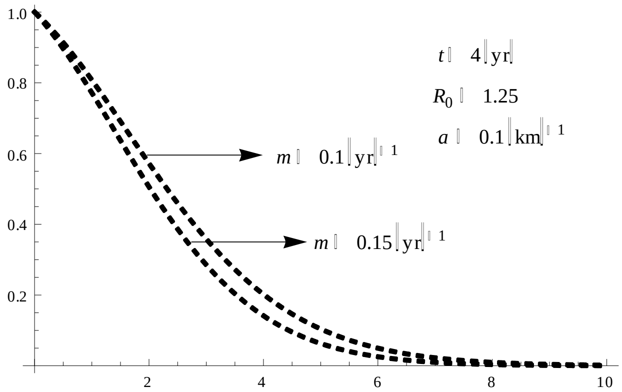

Fig 4. Solute transport described by the solution (37) where $t(\text{yr})=4$, $a=0.1(\text{km})^{-1}$ and $R_0=1.25$ for $m=0.1, 0.15(\text{year})^{-1}$

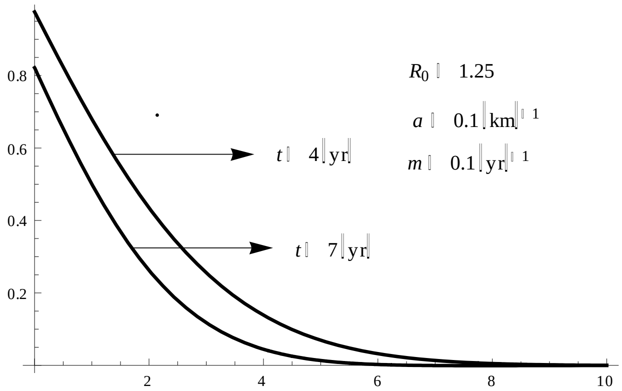

Fig 5. Solute transport described by the solution (47) where $m=0.1(\text{year})^{-1}$, $a=0.1(\text{km})^{-1}$ and $R_0=1.25$ for $t(\text{yr})=4, 7$

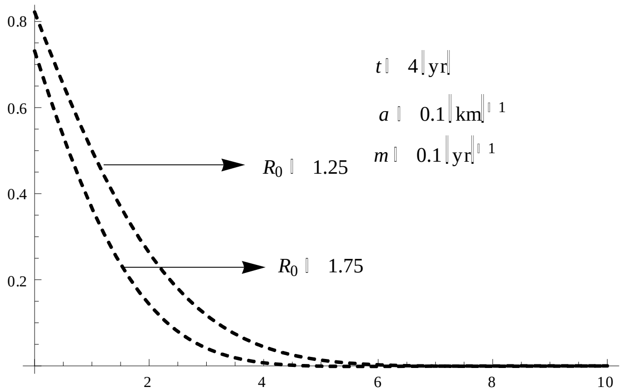

Fig 6. Solute transport described by the solution (47) where $m=0.1(\text{year})^{-1}$, $t(\text{yr})=4$ and $a=0.1(\text{km})^{-1}$ for $R_0=1.25, 1.75$

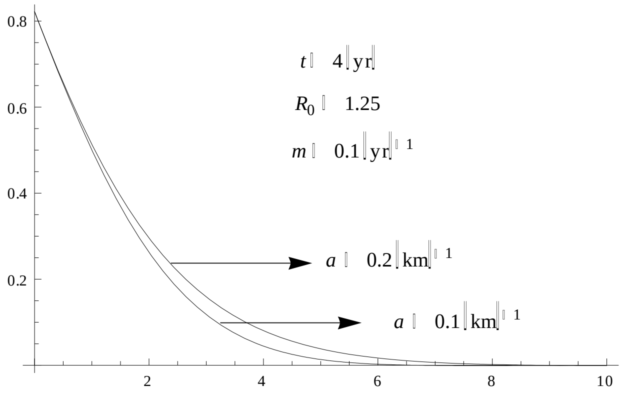

Fig 7. Solute transport described by the solution (47) where $m = 0.1(\text{year})^{-1}$, $t(\text{yr}) = 4$ and $R_0 = 1.25$ for $a = 0.1, 0.2\ (\text{km})^{-1}$

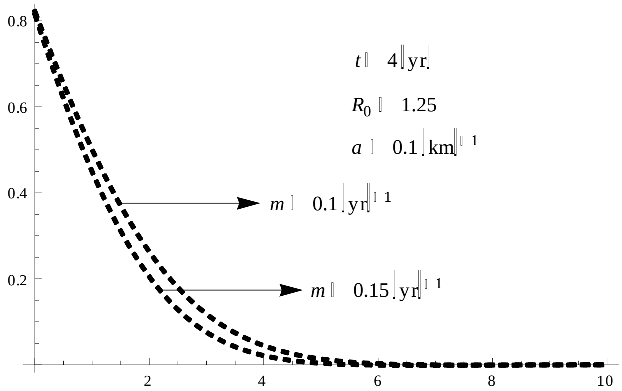

Fig 8. Solute transport described by the solution (47) where $t(\text{yr}) = 4$, $a = 0.1(\text{km})^{-1}$ and $R_0 = 1.25$ for $m = 0.1, 0.15(\text{year})^{-1}$

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