IMPLICIT ONE STEP CONTINUOUS HYBRID BLOCKS METHODS WITH FIVE OFF-STEPS POINT FROM FIFTH DEGREE CHEBYSHEV POLYNOMIALS

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ABSTRACT

In this paper a self starting one step continuous block hybrid formulae{CBHF} with fve off-steps points is developed from zeros of Chebyshev polynomial using collocation and interpolation techniques. The (CHBHF) is then used to produce multiple numerical integrators which are of uniform order and are arrange into a single block matrix equations. These equations are simultaneously applied to provide the approximate solution for the stiff ordinary differential equations. The order of method and stability of the block method is discussed and its accuracy is established. Furthermore, the new block method possesses the desirable features of being A-stable, which is requirement for a method to solve stiff differential equations, also being self-starting and eliminates the used of predictor-corrector method.

1.0 Introduction

Consider the initial value problem of ordinary differential equation possibly stiff ordinary differential equation (ODEs)

$$\varphi'(x) = f(x,\varphi) \qquad \varphi(x_o) = \varphi_0 \tag{1.1}$$

We seek a solution in the range $[a \le x \le b]$.where a and b are finite and we assume that if {1.1}satisfy the conditions which guarantee that the problem has a unique continuously Differentiable solution, which we indicated by $[\varphi(x)]$.

Consider the sequence of points $\{x_n\}$, $[x_n = a + nh, n = 0,1,2...\frac{b-a}{h}]$, where the parameter h is the step length and is constant. An essential property of the majority of computational methods for the solution of {1.1} is that of discritization, that is we seek an approximate solution, not on the continuous interval. $[a \le x \le b]$ .but on the discrete point set $\{x_n\}$

The k-step Linear Multistep method (LMM) for the solution of (1.1) is generally written as

$$\sum_{j=0}^{k} \alpha_j \varphi_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

Which has 2k+1 unknown $\alpha's$ *and* $\beta's$ and therefore can be of order 2k.But according to (Dahlquist, 1963).The order of equation (1.2) cannot exceed k+1, if (k is odd) or k+2 if (k is even) for the method to be stable. Several authors such as (Gear, 1965), (Stetter, 1964),and (Butcher, 1965) proposed the modified forms of (1.2) which were shown to overcome the Dahlquist barrier theorem .These methods,known as hybrid methods were obtained by incorporating off-step points in the derivation process.

$$\sum_{j=0}^{s}\alpha_j\phi_{n+j} = k\sum_{j=0}^{s+\mu}\beta_{\tau j} f_{n+\tau j} \quad (1.3)$$

Where $\mu \in [0\ 1]$ which were shown to be of order up to 2k+2.Howerever, (Gupta, 1978) note that deriving such kind of methods for (1.3) is more tedious due to the occurrence of the fractional off-step points, which increases the number of predictors needed to implement the method. The hybrid method proposed in this paper, does not share this disadvantage since it is self-starting .The algorithm was developed by Onumanyi et al (1994),Onumanyi & Sirisena, (1999) through continuous interpolants based on the work of Lie & Norsett (1986). (Lie, 2004).We also observe that block methods were first introduced by (Milne, 1953) for only as a means of obtaining starting values for predictor-corrector algorithms and has since then been develop by several researchers such as Chollom (2004),Fatunla (1994),Jator (2010).Also Chebyshev polynomials appear in many papers as a tool for approximating an ordinary differential equations such as paper by (Vigor-Agutar, 2006) where a Chebyshev polynomials where employ to obtain the internal stage for Lobatto IIIA implicit Runge-Kutta for the solution of stiff differential equations.T this end,our contionous representation generates a main discrete hybrid method which are combined and implemented as a block method,which simultanously generate approximations $\phi_{n+j\mu}$ to the exact solution $\phi(x_{n+j\mu})$ $j = 0,1,2,\ldots 5$,without loss of generality, the shifted second kind Chebyshev polynomials $U_n(x)$,which is given in terms of three terms recurrence relation in (Gautschi, 2004) as

$$U_{n+1}(x) = 2(2x-1)U_n - U_{n-1} \quad (1.4)$$

$$U_0(x) = 1, U_1 = 2(2x-1) \quad (1.5)$$

Therefore, for n=4 we've the fifth degree shifted second kind Chebyshev polynomials given as

$$U_5(x) := 1024x^5 - 2560x^4 + 2304x^3 - 896x^2 + 140x - 6 \quad (1.6)$$

Thus the eigenvalue of the characteristic equation of the companion matrix of (1.6), given as

$$\begin{pmatrix} \frac{2560}{1024} & -\frac{2304}{1024} & \frac{896}{1024} & -\frac{140}{1024} & \frac{6}{1024} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (1.7)$$

Where the eigenvalue of A are the zero's of (1.6) given as

$\mu_1 = \frac{1}{2} - \frac{1}{4}\sqrt{3}, \mu_2 = \frac{1}{4}, \mu_3 = \frac{1}{2}, \mu_4 = \frac{3}{4}, \mu_5 = \frac{1}{2} + \frac{1}{4}\sqrt{3}$ Which are incorporate as the off steps point into 1-step hybrid Adams-Moulton to obtain the block methods,

In this paper our aim is to generate a one-step implicit continuous hybrid block methods with three off-step points from zero's of (1.6) and to demonstrate the efficiency in its implementations on stiff problems.

2.0 Derivation of the Method

In this section, our objective is to derive the main hybrid block method of the form

$$\sum_{j=0}^{1} \alpha_j y_{n+j} = h \sum_{j=0}^{5} \beta_{\mu[j]} f_{n+\mu[j]} \quad (2.1)$$

Where α_j and $\beta_{\mu[j]}$ are unknown constant, also $\mu[j]$ are the zero's of (1.7) in the interval of $\mu \in [0,1]$.In order to obtain the unknown coefficients in (2.1) we now consider collocation polynomial Onumanyi et al (1994) of the form

$$\phi(\mathrm{x}) = \sum_{j=1}^{t+s-1} a_j x^j \quad (2.2)$$

Which can now be express inform of

$$\phi(\mathrm{x}) = \left\{ \sum_{j=1}^{t-1} a_{j,t+s-1} \phi_{n+j} + h \sum_{j=0}^{s-1} \beta_{j,t+s-1} f_{n+j} \right\} \left(1, x, x^2 \ldots x^{t+s-1}\right)^T \quad (2.3)$$

Thus, we can express equation (9) explicitly as follows

$$\phi(\mathrm{x}) = \left(\phi_n \ldots \phi_{n+t-1}, f_n, \ldots f_{n+s-1}\right) C^T \left(1, x, x^2 \ldots x^{t+s-1}\right)^T \quad (2.4)$$

Where

$$C = \begin{pmatrix} c_{1,1} & c_{1,t}\ldots\ldots & c_{1,t+s} \\ c_{2,1} & c_{2,t}\ldots\ldots & c_{2,t+s} \\ . & . & . \\ c_{t+s,1} & c_{t+s,t} & c_{t+s,t+s} \end{pmatrix} \quad (2.5)$$

Also

DC=I,

Therefore, $C = D^{-1}$

And

$$D = \begin{pmatrix} 1 & x_n\ldots\ldots & x_n^{t+s-1} \\ 1 & x_{n+1}\ldots\ldots & x_n^{t+s-1} \\ 0 & . & . \\ 0 & x_{n+s} & x^{t+s-2}{}_{t+s-1} \end{pmatrix} \quad (2.6)$$

The matrices (2.5) and (2.6) are of dimensions (t+s)X(t+s).We call the D the multistep collocation and interpolation matrix which has a very simple structure. It is similar to Vander monde matrix, consisting of distinct elements nonsingular, and of dimension (s+t) x(s+t) .This matrix affect the efficiency of (2.6).The choice of $C = D^{-1}$ leads to the determination of (2.6) is convergent with order p=t+s-1 of the constant coefficients $\alpha_{j,i+1}$ *and* $\beta_{j,i+1}$ .It was shown in (Dauda, 2011),that the method (2.6) is convergent with order p=t+s-1.We thus construct a k-step multistep method by imposing the following conditions.

$$\phi(\chi_{n+\mu j}),= \phi_{n+j}, \quad j = 0,1,2,\ldots.t-1 \quad (2.7)$$

$$\phi'(\chi_{n+\mu j}),= f_{n+\mu j}, \quad j = 0,1,2,\ldots.s-1 \quad (2.8)$$

Where $\mu \in [0,1], \phi(x_{n+\mu[j]}), f_{n+\mu[j]} = f(x_{n+\mu[j]})$ and n is the grid index. It should be noted that equation (2.7) and (2.8) leads to a system of t+s equations which must be solved to obtained the coefficients a_j algebraic computation, our method yields the expressing in the form

$$\phi(x) = \sum_{j=0}^{t-1} \alpha_j \phi_{n+j} + h \sum_{j=0}^{s-1} \beta_{\mu[j]} f_{n+\mu[j]} \qquad (2.9)$$

Which is used to generate the main discrete hybrid block method in the form of (3)

Now For k=1, taking the zero's of equation (1.8) and t=1, S=7, $x^i, i = 0,1,2,3,4,5,6$ and thus interpolating (2.9) at

$$x = \{x_n, x_{n+\mu 1}, x_{n+\mu 2}, x_{n+\mu 3}, x_{n+\mu 4}, x_{n+\mu 5}, x_{n+1}\} \qquad (2.10)$$

We generate the following continuous hybrid block method which can also be represented in Butcher Tableau as in concept of Runge-Kutta as follows

$$a = \begin{pmatrix} \varphi_n \\ \varphi_{n+\mu[1]} \\ \varphi_{n+\mu[2]} \\ \varphi_{n+\mu[3]} \\ \varphi_{n+\mu[4]} \\ \varphi_{n+\mu[5]} \\ \varphi_{n+\mu[1]} \end{pmatrix} \qquad (2.11)$$

$$b = \begin{pmatrix} \frac{1064+568\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{385+170\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{434+467\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{784+352\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{574+397\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{1855+810\sqrt{3}}{20160(-2+\sqrt{3})} & \frac{56-8\sqrt{3}}{20160(-2+\sqrt{3})} \\ \frac{1}{280} & \frac{296+189\sqrt{3}}{4032} & \frac{233}{2240} & \frac{-23}{1260} & \frac{23}{224} & \frac{296-189\sqrt{3}}{4032} & \frac{1}{280} \\ \frac{53}{2520} & \frac{16+7\sqrt{3}}{2520} & \frac{319}{1260} & \frac{41}{315} & \frac{-31}{1260} & \frac{16-7\sqrt{3}}{2520} & \frac{-17}{2520} \\ \frac{3}{280} & \frac{8+7\sqrt{3}}{448} & \frac{489}{2240} & \frac{39}{140} & \frac{279}{2240} & \frac{8-7\sqrt{3}}{448} & \frac{3}{280} \\ \frac{1064-568\sqrt{3}}{20160(2+\sqrt{3})} & \frac{-385-170\sqrt{3}}{20160(2+\sqrt{3})} & \frac{-434-467\sqrt{3}}{20160(2+\sqrt{3})} & \frac{-784-352\sqrt{3}}{20160(2+\sqrt{3})} & \frac{-574-397\sqrt{3}}{20160(2+\sqrt{3})} & \frac{-1855-810\sqrt{3}}{20160(2+\sqrt{3})} & \frac{56+8\sqrt{3}}{20160(2+\sqrt{3})} \\ \frac{1}{70} & \frac{8}{63} & \frac{8}{35} & \frac{82}{315} & \frac{8}{35} & \frac{8}{63} & \frac{1}{70} \end{pmatrix} \qquad (2.12)$$

$$c = \begin{pmatrix} f_n \\ f_{n+\mu[1]} \\ f_{n+\mu[2]} \\ f_{n+\mu[3]} \\ f_{n+\mu[4]} \\ f_{n+\mu[5]} \\ f_{n+1} \\ \end{pmatrix} \quad (2.13)$$

where the block is given as

$$a = bc \quad (2.14)$$

3.0 STABILITY ANALYSIS

It has been shown in (Chollom et al, 2007) that a block linear multistep method is said to be zero stable if the roots of

$$\rho(R) = \det\left\{\sum_{i=0}^{k} A^{+(i)} R^{k-i}\right\} = 0, R_j, \; j = 1(1)k \quad (3.1)$$

Of the first characterizes polynomials satisfies

$\left|R_j\right| \leq 1$ and the multiplicity must not exceed two, we then apply (3.1) to check for the zero stability of the derive block method in (3.1),after some simplification the roots of the block method is (0,0,0,0,0,0,1),which confirm the zero stability of block method.

3.1 Order and error constant

We adopt (Chollom et al, 2007) to obtain the order and error constant of the block method.

Which state that the block linear multistep method is said to be of order p if

$$\overline{c}_0 = \overline{c_1} = \overline{c_2} = \ldots \overline{c_p} = 0, \; \overline{c_{p+1}} \neq 0 \quad (3.2)$$

And the local truncation is expressed as

$$T_n = c_{p+1} h^{p+1} y^{p+1}(x_n) \quad (3.3)$$

Now to determine the order and error constant of the derived method we apply (3.2) for the order and (3.3) for truncation error, which we obtained as follow:

The integrators (2.14) is a one block one step hybrid methods of order $(7, 7, 7, 7,7,7)^{T}$ with error constants

$$c_{p+1} = c_7 = \{6.3073*10^{-10}, -3.5000*10^{-9}, 0, -3.5000*10^{-9}, -6.3070*10^{-10}, -1.1000*10^{-11}\}$$

3.2 Stability Function of the block method

The stability function of the block formula (2.14) are determine through the application to test equation

$$y' = \lambda y \quad \lambda < 0 \qquad (3.4)$$

Applying of (3.4) to (2.14) gives the stability function of the method as

$$R_5(z) = \frac{(2580480 + 1290240z + 291840z^2 + 38400z^3 + 3108z^4 + 146z^5 + 3z^6)}{(2580480 - 1290240z + 291840z^2 - 38400z^3 + 3108z^4 - 146z^5 + 3z^6)} \qquad (3.5)$$

It can be seen from (3.5) that the one step block method is A-stable in the spirit of (Brugnano, 1998),which requires that all $z = h\lambda \in C^-$ and $\mathrm{Re}(z) < 0$, $\rho(\eta, z)$ must have a dominant eigenvalue η_4 such that $|\eta_4| < 1$.from the analysis of the (2.14) we obtained the eigenvalue as (0,0,0, 0,0,0,0, η_4) and the dominant eigenvalue is the function in (3.5) of z.

3.3 Absolute stability region of the block method

To plot the region of absolute stability regions of (2.14),they are reformulated as General Linear Methods of (Burage, 1980) where they used a partition (s+t)X(s+t) matrix containing A,B,U,V express as in (Chollom & Ndam, 2007) where the absolute stability region of the method is defined as

$A = x \in C : \rho(\eta, z) = 1 \Rightarrow |\eta| \le 1$.In our on case, Maple is used to obtained the stability polynomials of the method as

$$\rho(\eta,z) = \frac{2(-1843200 - 161280z^2 + 921600z + 3840z^3 + 2232z^4 - 276z^5 + 11z^6)}{\left(1920 - 960z - 22z^3 + 204z^2 + z^4\right)\left(3840\eta - 1920\eta z - 44\eta z^3 + 408\eta z^2 + 2\eta z^4 - 3840 + 72z^2 - z^4\right)} \qquad (3.6)$$

The stability polynomial is used in matlab environment and produces the region of absolute stability of the block method as shown in figure 1.

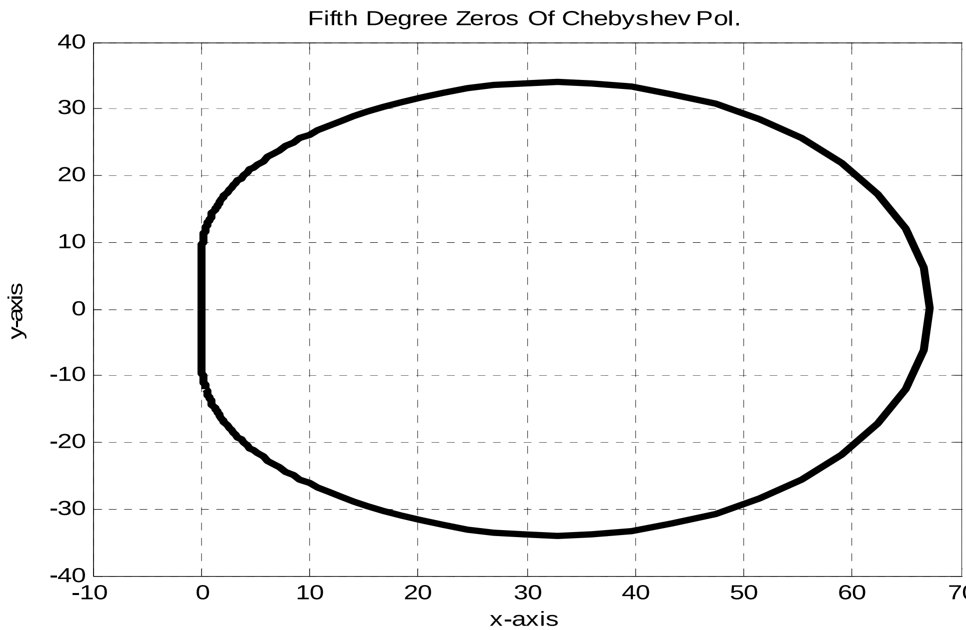

Figure 1: The Absolute Stability Region of the(BHAM) with zeros of Fifth Degree Chebyshev Polynomials. As an off-step points

The stability region for the method (2.14) lies outside the bounded region. Since all the region in the half plane is in the stability region, therefore the method is A-stable suitable for stiff problems.

4.0 Numerical Experiment

To check the numerical behavior when used to solve stiff initial –value problems, we now apply the derived methods to a variety of well-known problems, which have appeared different times in the literature, to ascertain the efficiency and accuracy of the derived methods and Also to measure the global error by obtaining the absolute error from the difference of the exact solution and the numerical solution.

Application:4.1 Prothero and Robinson Equation (Prothero & Robinson, 1974)

We consider the prothero-Robinson equation

$$\begin{aligned} &y'(t) = \lambda(y(t) - g(t)) + g'(t), \qquad t \in [0\ \ 10] \\ &y(0) = 0 \end{aligned} \qquad (4.1)$$

With $\lambda = -10^6$, $g(t) = \sin t$, and the exact solution by $y(t) = \sin t$,this problem constitute a stiff problem (since the eigenvalue is $\lambda = -10^6$)

No of steps (n)	x_n	Numer.Sol. Chev (five)	Exact Sol.	Abs. Err.
0	0	0	0	0
1	0.1	0.099833416646828	0.099833416646828	$2.776*10^{-16}$
2	0.2	0.198669330795061	0.198669330795061	$1.665*10^{-15}$
3	0.3	0.295520206661339	0.295520206661340	$3.333*10^{-15}$
4	0.4	0.389418342308650	0.389418342308651	$6.106*10^{-15}$
5	0.5	0.479425538604202	0.479425538604203	$9.999*10^{-15}$
6	0.6	0.564642473395034	0.564642473395035	$1.443*10^{-15}$
7	0.7	0.644217687237689	0.644217687237691	$1.999*10^{-15}$
8	0.8	0.717356090899520	0.717356090899523	$2.776*10^{-15}$
9	0.9	0.783326909627480	0.783326909627483	$4.882*10^{-15}$
10	1.0	0.841470984807892	0.841470984807897	$3.583*10^{-15}$

Table 1: Numerical Solution of (4.1).

Application 2.2. (Dauda, 2007)

$$y' = -20y + 20\sin x + \cos x \qquad y(0) = 1$$

Exact solution of application 3: $y(x) = e^{-20x} + \sin x$

No of steps (n)	x_n	Numer.Sol. Chev(five)	Exact Sol.	Abs. Err.
0		0	1.000000000000000	1.000000
1	0.1	0.235146948000019	0.235168699883441	$2.175*10^{-5}$
2	0.2	0.216979082562480	0.216984969683795	$5.887*10^{-6}$
3	0.3	0.297997763831441	0.297998958838006	$1.195*10^{-6}$
4	0.4	0.389753589318815	0.389753804936553	$2.156*10^{-7}$
5	0.5	0.479470902061465	0.479470938533966	$3.647*10^{-8}$
6	0.6	0.564648611685160	0.564648617607389	$5.922*10^{-8}$
7	0.7	0.644218517832015	0.644218518766410	$9.343*10^{-9}$
8	0.8	0.717356203290861	0.717356203434698	$1.438*10^{-10}$
9	0.9	0.783326924836309	0.783326924857463	$2.115*10^{-11}$
10	1.0	0.841470986866675	0.841470986869050	$2.375*10^{-12}$

Table 2: Numerical Solution of (5.2). For the Method N=1

5.0 Conclusion

A continuous block hybrid method with three off-step points from zeros of second kind Chebyshev polynomials has been proposed and implemented as a self starting method in block form for solving stiff differential equations (SDE) .The method has proving to be very competitive with other well known method with just one step and few off-step point, we have also shown the accuracy of the block method in considering the tables shown in table 1 and 2.

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